Robust random search with scale-free stochastic resetting

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A new model of search based on stochastic resetting is introduced, wherein rate of resets depends explicitly on time elapsed since the beginning of the process. It is shown that rate inversely proportional to time leads to paradoxical diffusion which mixes self-similarity and linear growth of the mean-square displacement with nonlocality and non-Gaussian propagator. It is argued that such resetting protocol offers a general and efficient search-boosting method that does not need to be optimized with respect to the scale of the underlying search problem (e.g., distance to the goal) and is not very sensitive to other search parameters. Both subdiffusive and superdiffusive regimes of the mean-squared displacement scaling are demonstrated with more general rate functions.

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I. INTRODUCTION

Any search is intrinsically governed by randomness: the need to perform it underscores the ignorance of the searching agent and its inability to predict the outcome. In particular, the agent may not know in advance how much time, on average, it will take to find a target and how much variability in search time is expected due the randomness. How can the efficiency of a search strategy be boosted without acquiring such information beforehand? It may happen that the longer the search lasts, the more unfavorable random instance the agent is experiencing, and thus the longer the expected remaining time to find the target. It is then true that restarting the process anew may be beneficial, as it lowers the expected completion time. It has been shown that this is indeed the case if the coefficient of variation of completion times is larger than 1 [1]. This conditions holds if the completion times of a search process are drawn from the hyperexponential or power-law distributions. Such situation arises, for instance, in the one-dimensional, unbiased Brownian motion with a single target. The possibility of arbitrarily long excursions far from the target manifests itself in large fluctuations and the associated power-law distribution of the first passage times.

In the engineering community such mechanism was studied in the context of computer software systems [2–5] and nonconvex optimization methods [6–9]. More recently, Evans and Majumdar introduced a model with time-homogeneous stochastic resetting (i.e., with exponentially distributed waiting times between the resets) in tandem with diffusion [10]. They showed that one-dimensional [10] and multidimensional [11] diffusion with resets exhibit finite mean first passage times (MFPTs) which can be optimized with respect to resetting rate r. Moreover, resets lead to nonequilibrium steady-states with a combination of local and nonlocal currents [10,12,13]. The simplicity and nontrivial behavior of these systems has sparked the interest of the physics community, with a considerable amount of research focusing on modifying the model to include nontrivial boundary conditions [14–16], external potentials [17], anomalous transport [12,18–24], drift [21,25], delays following the resets [1,26–28], and nonexponential distributions of waiting times between resets [13,29,30]. Resets were also studied in the context of stochastic energetics [31], enzymatic reactions [32], fluctuating interfaces [33], and power-law distributions in nonequilibrium systems [34].

Nonexponential waiting times between the resets can be generated by time-dependent resetting rates [29,30]. In models introduced in these previous works, rate depends on the time elapsed since the last reset. Here we introduce a new model with resetting rate $r(t)$ that depends on the time elapsed since the start of the process. Such nonstationary resetting protocol, while preserving the Markovian character of the dynamics, introduces time-inhomogeneity. As we will show, this dramatically affects the behavior of the process. If $r(t)$ decays with time, then the process does not converge to any steady state, with the mean-square displacement (MSD) growing indefinitely. Moreover, $r(t)$ that is inversely proportional to time provides an efficient and robust mechanism to boost the time efficiency of a search. It has previously been shown that deterministic resetting is the optimal way of minimizing search time via resets [35]. However, similarly to stochastic resetting with a constant rate, such resetting has to be optimized assuming full knowledge of details of a given search problem. In contrast, the resetting mechanism we propose does not have to be adapted to the timescale of the search problem.

II. DIFFUSION WITH TIME-DEPENDENT RESETTING

We start from one-dimensional diffusion as the underlying search process, which will help build the intuition. Later we present a more general framework. Since the process of diffusion with memoryless resetting is Markovian, it is fully characterized by the initial position distribution and the propagator $\rho(x, t|x_i, t_i)$, which denotes the probability density function of x at time t of a particle that at time $t_i < t$ was at x_i. The particle diffuses till a resetting event that brings it instantaneously back to the resetting position x_0 (equal to the position of the particle at time $t = 0$) [36]. The particle then continues to diffuse until the next reset. The corresponding propagator solves the following partial differential equation:

$$\partial_t \rho(x, t|x_i, t_i) = D\partial_{xx}^2 \rho(x, t|x_i, t_i) - r(t)\rho(x, t|x_i, t_i) + r(t)\delta(x - x_0), \quad (1)$$

where ∂_z denotes the partial derivative with respect to z and D is the diffusion coefficient. Such process has been analyzed extensively in the case of a constant rate $r(t) = r$, with the corresponding resets described by the homogeneous Poisson point process. It has been shown [10] that this process, in the absence of targets, attains a nonequilibrium steady state. If a single target, represented by an absorbing boundary, is placed at the origin, then there exists an optimal rate $r^* \propto x_0^{-2}$ at which the MFPT is minimized.

A. Scale-free resetting

In this work we mainly focus our attention on the special case of

$$r(t) = \frac{\alpha}{t}. \quad (2)$$

The only parameter α is dimensionless—this form of resetting does not introduce any timescale. For this reason, we refer to Eq. (2) as a *scale-free resetting*. Resets described by Eq. (2) are generated by the inhomogeneous Poisson point process with an average intensity (expected number of events) within a time period $(t_0, t_1]$ given by

$$N(t_0, t_1) = \alpha \ln \frac{t_1}{t_0}. \quad (3)$$

Note that our choice of $r(t)$ features a singularity at $t = 0$, which translates into diverging intensity with $t_0 \to 0$. Any practical applications of scale-free resetting have to introduce a short-term cutoff to avoid resetting with infinite frequency. However, all our theoretical predictions without an explicit account of the cutoff are still valid at times much longer than the cutoff.

As evident from Eq. (1), the process Eq. (1) with Eq. (2) preserves the self-similarity of the pure Brownian motion, i.e., the rescaling $x \to cx$ and $t \to c^2 t$ does not change any observables (see Fig. 1). We can therefore expect that the mean-square displacement (MSD) scales linearly with time. Indeed, straightforward calculations [multiply Eq. (1) by $(x - x_0)^2$, integrate over x and solve the resulting ordinary differential equation] lead to

$$\langle (x(t) - x_0)^2 \rangle = \int_{-\infty}^{\infty} dx x^2 \rho(x, t|x_0, 0) = \frac{2Dt}{1 + \alpha}. \quad (4)$$

Although the self-similarity and associated linear time-dependence of the MSD bear a strong resemblance to free diffusion, resets vastly modify other statistics. For instance, the Fourier transform of the propagator reads

$$\rho(k, t|x_0, 0) = \alpha e^{ikx_0} t^{-\alpha} e^{-Dk^2 t} \int_0^t d\tau \tau^{\alpha - 1} e^{Dk^2 \tau}. \quad (5)$$

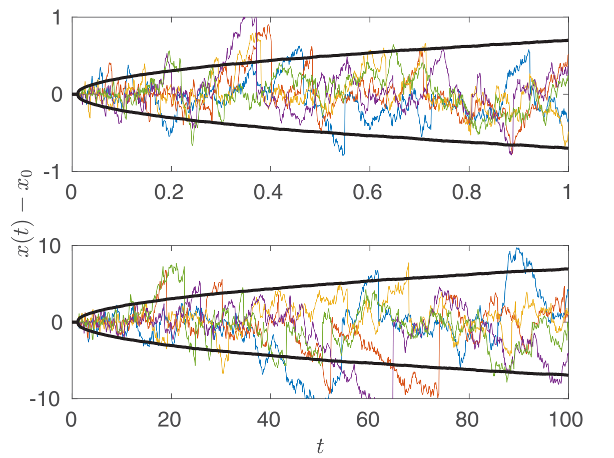

FIG. 1. Sample trajectories (colored, thin lines) and percentile lines (thick, black lines) of diffusion with scale-free resetting with $\alpha = 10$. A comparison of short (top) and long (bottom) timescales illustrates the self-similarity of the process. The presented percentile lines $x_q(t)$ are such that together only 10% of the sample trajectories have $x(t) > x_q(t)$ (upper branch) or $x(t) < x_q(t)$ (lower branch).

This propagator is clearly non-Gaussian and has a cusp at $x = x_0$. It takes a particularly simple form for $\alpha = 1$:

$$\rho(k, t|x_0, 0) = e^{ikx_0} \frac{1 - e^{-Dk^2 t}}{Dk^2 t}. \quad (6)$$

Similar non-Gaussian displacement distributions with a cusp, accompanied by a linear time dependence of the MSD have been observed in several systems of diffusing colloidal particles [37,38]. Such systems have recently attracted considerable attention among theoreticians, who since then have developed multiple models with fluctuating diffusivity [37,39–42]. Scale-free resetting provides an alternative mechanism behind such behavior.

B. Generalizations of the resetting rate function

In addition to the presented applications to search problems, a generalization of the proposed resetting mechanism can be applied to model anomalous transport phenomena. The MSD of Eq. (1) with $r(t) = \alpha/t^\mu$ is given by the formula

$$\langle (x(t) - x_0)^2 \rangle = 2D \exp\left(-\frac{\alpha}{1 - \mu} t^{1-\mu}\right) \times \int_0^t d\tau \exp\left(\frac{\alpha}{1 - \mu} \tau^{1-\mu}\right), \quad (7)$$

which for $\mu < 1$ exhibits a smooth transition between diffusive behavior with the MSD $\approx 2Dt$ for $t \ll \tau_\alpha$ and subdiffusive behavior with the MSD $\approx 2Dt^\mu/\alpha$ for $t \gg \tau_\alpha$, where $\tau_\alpha = \alpha^{-1/(1-\mu)}$ is the timescale introduced by the power-law resetting. In the opposite case of $\mu > 1$ the long-term behavior is diffusive, whereas at short times an apparent superdiffusivity with the MSD $\approx 2D\tau^\mu/\alpha$ is observed. By assuming $r(t) \propto (\log t)^{-1}$ one can also model an ultra-slow diffusion with the MSD $\propto \log t$, a behavior previously uncovered in the strongly non-Markovian random walks with preferential relocations to places visited in the past [43].

One could involve heavy-tails jump distributions leading to a competition between superdiffusivity of the Lévy flights and subdiffusive tendency from resets, similar to the competition observed in the continuous-time random walk scheme [44–46]. Many different variants of a combination of anomalous transport with constant-rate stochastic resetting were previously studied [12,18–21,24] and were shown to exhibit nontrivial features, including first and second order transitions of the optimal search [18,19,21,47] and a nonmonotonic behavior of the MSD [24]. For this reason, we expect that a combination of scale-free resets with independent constant-rate resets may lead to thought-provoking phenomena.

III. COMPLETION TIMES

In the previous section we have shown how time-dependent resetting affects the time evolution and shape of the displacement distribution. Another striking consequence of scale-free resetting can be observed in the statistics of the first passage times. As we show in the following, in contrast to free diffusion on infinite line, diffusion with scale-free resetting can find a target in a finite MFPT. Moreover, due to the self-similarity, we can expect that the MFPT is proportional to x_0^2, where x_0 denotes the initial position with respect to the target.

This observation prompted us to explore more general scenarios of search under scale-free resetting beyond diffusion on infinite line. Since the following calculations apply to general search times that may not be generated as the first passage times of some stochastic process, we hereafter use the term *completion times*. We show that scale-free resetting is *robust*, i.e., the optimal form of the scale-free resetting (as prescribed by the parameter α) does not depend on the timescale. Thus, the protocol can be optimized in a parsimonious manner, without knowing how much time on average the underlying search takes.

A. General analysis

Instead of assuming that the underlying search process is described by a simple, one-dimensional diffusion, we first consider any random search process. Let a nonnegative random variable T_0 be the completion times of the original process without resetting. Similarly, let T_α denote the completion times of the same process modified by introducing scale-free resets of the form Eq. (2). Our goal is to calculate statistics of T_α given known statistics of T_0. In particular, we will be interested in the mean completion time (MCT), denoted as $\overline{\mathrm{T}}_\alpha$. For any absolute-time-dependent rate function $r(t)$, one can link the survival probability of the model with resets $\mathrm{S}_\alpha(t)$ with the survival probability of the underlying reset-free process $\mathrm{S}_0(t)$ by means of the following renewal equation [48]:

$$\mathrm{S}_\alpha(t) = \mathrm{S}_0(t)\Psi(0,t) + \int_0^t d\tau\, r(\tau)\,\mathrm{S}_\alpha(\tau)\,\mathrm{S}_0(t-\tau)\Psi(\tau,t), \tag{8}$$

where

$$\Psi(t_0,t_1) = \exp\left(-\int_{t_0}^{t_1} d\tau\, r(\tau)\right) \tag{9}$$

is the probability of no resets in the interval of time $(t_0, t_1]$. Equation (8) splits the survival probability into two cases: either there are no resets until time t, which happens with probability $\Psi(0,t)$, or there are resets in this time period with the last reset at time τ. The integral appears here because τ can be anywhere between 0 and t. In the case of scale-free resetting Eq. (2), the no-reset probability is given by

$$\Psi_\alpha(t_0,t_1) = \left(\frac{t_0}{t_1}\right)^\alpha. \tag{10}$$

Combining the general expression Eq. (8) with Eqs. (10) and (2) we arrive at the equation

$$\mathrm{S}_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t d\tau\, \tau^{\alpha-1}\,\mathrm{S}_\alpha(\tau)\,\mathrm{S}_0(t-\tau). \tag{11}$$

To solve Eq. (11) we introduce an auxillary function,

$$\mathrm{g}_\alpha(t) \equiv t^{\alpha-1}\,\mathrm{S}_\alpha(t), \tag{12}$$

which solves a simpler integral equation,

$$\mathrm{g}_\alpha(t) = \frac{\alpha}{t}\int_0^t d\tau\, \mathrm{g}_\alpha(\tau)\,\mathrm{S}_0(t-\tau). \tag{13}$$

By denoting $\mathcal{L}\{f(t)\} \equiv \tilde{f}(s)$ as the Laplace transform of $f(t)$, the corresponding Laplace-transformed equation reads

$$\partial_s\, \tilde{\mathrm{g}}_\alpha(s) = -\alpha\, \tilde{\mathrm{S}}_0(s)\,\tilde{\mathrm{g}}_\alpha(s). \tag{14}$$

In the following we will assume that the probability of finding the target in no-time is equal to zero. In this case the corresponding survival probability can be written in the form [49]

$$\tilde{\mathrm{S}}_0(s) = \frac{1-\tilde{\rho}_0(s)}{s}, \tag{15}$$

where $\rho_0(t)$ is the probability density function of the completion time of the process without resetting, and $\lim_{s\to\infty}\tilde{\rho}_0(s) = 0$. We can easily solve the differential Eq. (14), leading to

$$\tilde{\mathrm{g}}_\alpha(s) = C_\alpha s^{-\alpha}\exp\left(-\alpha\int_s^\infty du\frac{\tilde{\rho}_0(u)}{u}\right), \tag{16}$$

where C_α is a constant yet to be determined.

We now show how to recover $\tilde{\mathrm{S}}_\alpha(s)$ from $\tilde{\mathrm{g}}_\alpha(s)$. We employ the identity

$$\int_s^\infty du(u-s)^\beta e^{-st} = \Gamma(\beta+1)t^{-1-\beta}e^{-st}, \tag{17}$$

which holds for $\beta > -1$ [50]. From Eq. (17) we deduce that

$$\mathcal{L}\{t^{-\beta-1}f(t)\} = \frac{1}{\Gamma(\beta+1)}\int_s^\infty du(u-s)^\beta \tilde{f}(s). \tag{18}$$

We can apply Eq. (18) in Eq. (12) by replacing β with $\alpha-1$ and noticing that $\mathcal{L}\{tf(t)\} = -\partial_s\tilde{f}(s)$ [51], which leads to

$$\tilde{\mathrm{S}}_\alpha(s) = -\frac{1}{\Gamma(\alpha)}\int_s^\infty du(u-s)^{\alpha-1}\partial_u\,\tilde{\mathrm{g}}_\alpha(u). \tag{19}$$

In the last step we plug Eq. (16) into Eq. (19). The initial condition

$$\lim_{t\to 0}\mathrm{S}_\alpha(t) = \lim_{s\to\infty} s\,\tilde{\mathrm{S}}_\alpha(s) = 1 \tag{20}$$

allows us to calculate $C_\alpha = \Gamma(\alpha)$. The final result reads

$$\tilde{\mathrm{S}}_\alpha(s) = \alpha \int_s^\infty du \left(\frac{u-s}{u}\right)^{\alpha-1} \frac{1-\tilde{\rho}_0(u)}{u^2} \times \exp\left(-\alpha \int_u^\infty dv \frac{\tilde{\rho}_0(v)}{v}\right). \tag{21}$$

Equation (21) allows us to calculate any statistics of the completion times, at least in principle. In particular, the MCT is simply given by $\tilde{\mathrm{S}}_\alpha(0)$, i.e.,

$$\overline{\mathrm{T}}_\alpha = \alpha \int_0^\infty du \frac{1-\tilde{\rho}_0(u)}{u^2} \exp\left(-\alpha \int_u^\infty dv \frac{\tilde{\rho}_0(v)}{v}\right). \tag{22}$$

B. Special cases

Here we present two special cases of the search process T_0, which illustrates basic features of the scale-free resetting.

1. One-dimensional diffusion with a single trap

In the first example we consider one-dimensional diffusion Eq. (1) with a single target (trap). The probability density function of the completion times (here corresponding to the first passage times) for $\alpha = 0$ (no resets) is well-known and in the Laplace space reads [49]

$$\tilde{\rho}_0(s) = e^{-\sqrt{s\tau_{\mathrm{diff}}}}, \tag{23}$$

where $\tau_{\mathrm{diff}} \equiv x_0^2/D$. Importantly, $\rho_0(t) \sim t^{-3/2}$ for large t and thus the MCT is infinite. Plugging Eq. (23) into the general expression for the MCT (22) leads to a rather complicated integral, which can nevertheless be easily integrated numerically. Numerical simulations confirm our theoretical prediction (cf. Fig. 2) and show that the MCT attains its minimum value of $\overline{\mathrm{T}}_{\alpha^*}/\tau_{\mathrm{diff}} \approx 1.97$ at $\alpha^* \approx 3.5$. Moreover, as expected from the dimensionality analysis, the MCT scales quadratically with the initial distance to the target.

We can learn a lot about the distribution of T_α by directly analyzing the auxiliary function Eq. (16), which in the case of diffusion takes the form

$$\tilde{g}_\alpha(s) = \Gamma(\alpha+1)s^{-\alpha} \exp\left[-2\alpha E_1\left(\sqrt{s\tau_{\mathrm{diff}}}\right)\right], \tag{24}$$

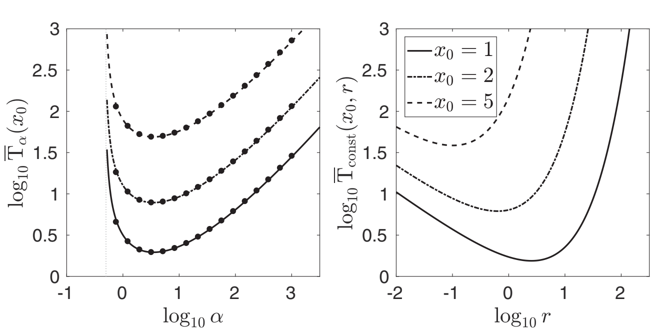

FIG. 2. The MCT of diffusion with scale-free resetting as a function of α (left) and the MCT of diffusion with constant-rate resetting as a function of r (right). The gray vertical line denotes the asymptote at $\alpha = \frac{1}{2}$. Dots denote simulation results which were obtained by means of the Euler-Maruyama stochastic integration method with a bias reduction technique that modifies the stopping rule close to the target [52] with $D = 1$, integration time step $\Delta t = 10^{-3}$, and number of samples $M = 10^6$. Notice log-log scale.

where

$$E_1(x) = \int_x^\infty dt \frac{e^{-t}}{t} \tag{25}$$

is a version of the exponential integral. For $x > 0$ it can be expressed as

$$E_1(x) = -\gamma - \ln x - \sum_{k=1}^\infty \frac{(-x)^k}{k \cdot k!}, \tag{26}$$

where $\gamma \approx 0.5772$ stands for the Euler-Mascheroni constant. Thus the solution Eq. (24) has the following representation:

$$\tilde{g}_\alpha(s) = \Gamma(\alpha+1)\tau_{\mathrm{diff}}^\alpha \exp\left(2\alpha\gamma + 2\alpha \sum_{k=1}^\infty \frac{(-\sqrt{s\tau_{\mathrm{diff}}})^k}{k \cdot k!}\right). \tag{27}$$

By letting $s = 0$ in Eq. (27) we arrive at a simple formula for the αth (fractional) moment,

$$\left\langle \mathrm{T}_\alpha{}^\alpha \right\rangle = \alpha \lim_{s\to 0} \tilde{g}_\alpha(s) = \Gamma(\alpha+1)e^{2\alpha\gamma}\tau_{\mathrm{diff}}^\alpha. \tag{28}$$

For $\alpha = 1$ we obtain a simple expression for the MCT,

$$\overline{\mathrm{T}}_1(x_0) = e^{2\gamma}\tau_{\mathrm{diff}} \approx 3.17x_0^2/D. \tag{29}$$

As expected, the MCT scales quadratically with the distance to the target. Small s expansion of (27) shows that $g_\alpha(t) \sim t^{-3/2}$ for large t, which together with Eq. (12) implies that the tail of the survival probability behaves like

$$\mathrm{S}_\alpha(t) \sim t^{-1/2-\alpha}. \tag{30}$$

Therefore, the fractional moments $\langle \mathrm{T}_\alpha{}^\nu \rangle$ are finite if and only if $\nu < \alpha + 1/2$. In particular, the MCT is finite for $\alpha > 1/2$; see Fig. 2.

2. Random search with failure

As we show in Sec. III C, the optimal choice of α does not depend on the search-problem timescale. However, this choice may be sensitive to other features of ρ_0. In the second example we explore the resilience of the scale-free resetting to search failures. Let the search problem be described by the density

$$\rho_0(t) = p\delta(t-\tau_0), \tag{31}$$

where $p \leqslant 1$ is the probability of a successful search. The trivial case $p = 1$ corresponds to deterministic completion time, whereas for $p < 1$ the density is unnormalized as there is nonzero probability $1-p$ that the search ends with a failure; i.e., the completion time is infinite. We plug

$$\tilde{\rho}_0(s) = pe^{-s\tau_0} \tag{32}$$

into Eq. (16) and arrive at

$$\tilde{g}_\alpha(s) = \Gamma(\alpha+1)s^{-\alpha} \exp[-\alpha p E_1(s\tau_0)], \tag{33}$$

and thus via the expansion Eq. (26),

$$\mathrm{S}_\alpha(t) \sim t^{-\alpha p}, \tag{34}$$

for $p < 1$ and large t. We conclude that the MCT is finite if and only if $p = 1$ or $\alpha > 1/p$; see Fig. 3. Moreover, the MCT is again proportional to timescale τ_0:

$$\overline{\mathrm{T}}_\alpha(p) = \tau_0\alpha \int_0^\infty ds \frac{1-pe^{-s}}{s^2} e^{-\alpha p E_1(s)}. \tag{35}$$

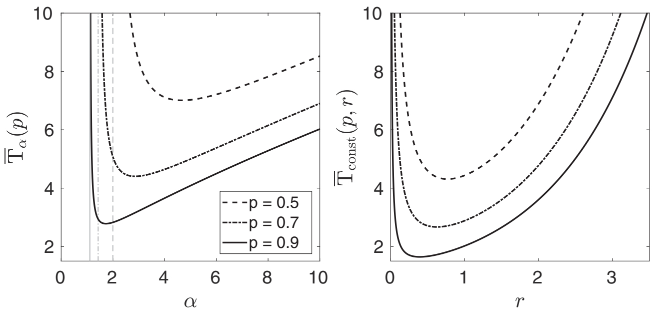

FIG. 3. The MCT of the failure model Eq. (31) with scale-free resetting as a function of α, Eq. (35) (left) and with constant-rate resetting as a function of r, Eq. (42) (right). Gray vertical lines denote asymptotes given by $\alpha_\infty = 1/p$.

This example illustrates the fact that, while robust to the timescale, scale-free resetting may be sensitive to other features of the search problem. In particular, the higher the probability of the search failure, the larger resetting intensity α is necessary to assure a finite value of the MCT. We discuss this issue in more detail in Sec. III D 3.

C. Linear scale-dependence

We have seen that in two simple special cases the MCT is proportional to the timescale of the underlying search problem. Here we show that this is always the case and that this observation generalizes to higher moments. Given the completion time of a process without resets T_0, we define a rescaled completion time as $\tau_0\,\mathrm{T}_0$. As in Sec. III B 2, the parameter τ_0 denotes the timescale of the search problem, and may be related to its different features, e.g., in the case of diffusion τ_0 is proportional to x_0^2/D. The corresponding probability density functions are related as

$$\tilde{\rho}_0(s,\tau_0) = \langle \exp(-s\tau_0\,\mathrm{T}_0)\rangle_{\mathrm{T}_0} = \tilde{\rho}_0(\tau_0 s, 1), \qquad (36)$$

whereas the survival probabilities

$$\tilde{\mathrm{S}}_0(s,\tau_0) = \tau_0\,\tilde{\mathrm{S}}_0(\tau_0 s, 1). \qquad (37)$$

Let $\mathrm{T}_\alpha(\tau_0)$ denote the completion time of a process with scale-free resetting with the underlying reset-free completion times given by $\tau_0\,\mathrm{T}_0$. We substitute variables $v \to v/\tau_0$ and $u \to u/\tau_0$ in Eq. (21) and obtain $\tilde{\mathrm{S}}_\alpha(s,\tau_0) = \tau_0\,\tilde{\mathrm{S}}_\alpha(\tau_0 s, 1)$, which leads to the conclusion that

$$\mathrm{T}_\alpha(\tau_0) \sim \tau_0\,\mathrm{T}_\alpha(1), \qquad (38)$$

where $\sim$ denotes the equality of distributions. For all finite fractional moments

$$\langle \mathrm{T}_\alpha(\tau_0)^\nu\rangle \propto \tau_0^\nu, \qquad (39)$$

and thus the mean value scales linearly $\overline{\mathrm{T}}_\alpha(\tau_0) \propto \tau_0$: scale-free restarts of the form Eq. (2) yield the MCT that is proportional to timescale of the underlying search problem, which explains why in the case of diffusion $\overline{\mathrm{T}}_\alpha \propto x_0^2$. As a consequence of Eq. (39), the optimal value of α, as defined by any moment, is robust to changes of τ_0. To see this, we can rewrite Eq. (39) in the form $\langle \mathrm{T}_\alpha(\tau_0)^\nu\rangle = \tau_0^\nu h(\alpha)$, where the function $h(\alpha)$ does not depend on τ_0. Clearly, the value of α that minimizes $h(\alpha)$ at the same time minimizes the $\langle \mathrm{T}_\alpha(\tau_0)^\nu\rangle$ for any nonzero value of τ_0.

This result is quite remarkable: the scale-free resetting provides a parsimonious search boosting technique that does not employ any knowledge about the timescale of the underlying search problem.

D. Comparison with other resetting protocols

1. Resetting protocols

To understand merits and limitations of the proposed scale-free resetting, it is instructive to compare it to other *resetting protocols*, i.e., predefined schemes of introducing resets into the system. We focus solely on (possibly nonstationary) feed-forward protocols that are independent from the state of the system [53].

Due to its simplicity, constant-rate resetting [i.e., $r(t) = r$] is particularly popular in the literature. Diffusion with such resetting has been studied extensively and its optimal MCT is given by $\overline{\mathrm{T}}_{\rm const}(x_0, r^*) \approx 1.54x_0^2/D$ with the optimal resetting rate$r^* \propto x_0^{-2}$ [54].

Another protocol that is of interest to us is a periodic, deterministic resetting. Following Refs. [1,35] we call this resetting *sharp*. Sharp resetting is important, as it was shown to form the dominant strategy in the stationary setting [35].

2. Fluctuating or unknown environment

Although the constant resetting can work better than the scale-free resetting [$\overline{\mathrm{T}}_{\rm const}(x_0, r^*) < \overline{\mathrm{T}}_{\alpha^*}(x_0)$], the optimal scale-free search parameter α^* does not involve any knowledge about the distance to the target. In contrast, the optimal constant rate r^* depends strongly on x_0; see Fig. 2. Hence, optimizing r may be hard, or even impossible, if the position of the target is unknown.

Consider a scenario of a single, immobile target placed at a random position and assume the location of the target does not change between the resets, but is drawn independently for separate trials [19,55,56]. This is equivalent to the resetting (and initial) position being drawn from a distribution $\rho_X(x_0)$, i.e., the MCTs of the scale-free and constant resetting processes in this case can be calculated as $\langle \overline{\mathrm{T}}_\alpha(x_0)\rangle_{x_0\sim\rho_X}$ and $\langle \overline{\mathrm{T}}_{\rm const}(x_0)\rangle_{x_0\sim\rho_X}$, respectively. The distribution ρ_X may represent the real variability of the environment or the ignorance of the searching agent.

As an important special case, let us take the Laplace distribution $\rho_X(x_0) = \exp(-|x_0|/\lambda)/(2\lambda)$, which maximizes entropy for a given average distance to the target $\lambda = \langle |x_0|\rangle$. In this case, the MCT of diffusion with constant resetting rate r reads

$$\langle \overline{\mathrm{T}}_{\rm const}(x_0, r)\rangle_{x_0\sim\rho_X} = \frac{\lambda}{\sqrt{Dr} - r\lambda}, \qquad (40)$$

with the minimum value of $4\lambda^2/D$ at $1/r^* = 4\lambda^2/D$. In contrast, since the variance of the Laplace distribution is equal to $2\lambda^2$, the MCT of diffusion with scale-free resetting is given by

$$\langle \overline{\mathrm{T}}_\alpha(x_0)\rangle_{x_0\sim\rho_X} = f(\alpha)\frac{2\lambda^2}{D}, \qquad (41)$$

where $f(\alpha) = \overline{\mathrm{T}}_\alpha(x_0)/\tau_{\mathrm{diff}}$ is the same prefactor as in the case of constant x_0 (Fig. 2) with the minimum $f(\alpha^*) \approx 1.97$ at $\alpha^* \approx 3.5$—in this case the scale-free resetting leads to a (slightly) better efficiency than the constant resetting. More importantly, in that case the optimal choice of α does not depend on λ, to which the optimal constant rate is quite sensitive. Indeed, if the chosen constant rate is larger than $r = D/\lambda^2$, the MCT diverges; see Eq. (40).

More broadly, the MCT of constant resetting search diverges for any distribution of x_0 that has tails heavier than exponential, in particular for any power-law distribution. In contrast, the MCT of diffusion with scale-free resetting remains finite for any distribution of x_0 with finite variance and the choice of the optimal parameter α does not depend on any aspect of the distribution of x_0,

3. Trade-off

Our results suggest that resetting protocols entail a natural trade-off between sensitivity of the protocol outcomes to two distinct features of the completion times of the underlying search problem: its timescale and the shape of its tail. It is convenient, in this context, to analyze the survival probability of the completion times. The tail of the survival probability captures both probability of large fluctuations and the failure probability, i.e., in the case of a nonzero failure probability, the survival probability does not decay to zero.

The scale-free resetting is robust to the timescale. However, the timescale robustness does not come without a price: due to the ever-growing intervals between succeeding resets, this protocol is rather sensitive to the tail of the distribution. In the case of a one-dimensional diffusive search, the survival probability of the completion times has a power-law tail $\sim t^{-1/2}$. The scale-free resetting to some extent tempers the tail, leading to a finite value of the MCT for $\alpha > \frac{1}{2}$. However, the fluctuations of the completion times are still large, with the survival probability $\sim t^{-1/2-\alpha}$. Thus, the variance diverges for $\alpha \leqslant \frac{3}{2}$. Similarly, the scale-free resetting is sensitive to the failure probability: as shown above, to retain a finite MCT, α has to be larger than $1/p$, where $1 - p$ is the failure probability; see Fig. 3.

At the other extreme is the sharp resetting protocol. Indeed, the optimal sharp resetting shows shorter MCTs and lower relative fluctuations than any stochastic resetting in a known search environment [35]. Additionally, it is easy to show that the optimal sharp protocol does not depend on the failure probability p. However, sharp resets are at the same time extremely sensitive to the timescale of the search process. In the case of diffusion they lead to divergent MCTs for any Laplace distribution of x_0 [55].

Within the framework of the discussed trade-off, constant-rate resetting is placed in between sharp resetting and scale-free resetting. In the case of diffusion, fluctuations of the completion times are strongly reduced by this kind of resetting—all moments are finite for any r [1,35]. In the simple case of constant time failure model Eq. (31) its MCT reads

$$\overline{\mathrm{T}}_{\mathrm{const}} = \frac{1}{r}\left(\frac{e^{r\tau_0}}{p} - 1\right). \tag{42}$$

The optimal parameters of the constant-rate resetting protocol depend both on scale and failure probability; see Fig. 3. On the one hand, in contrast to the scale-free resetting protocol, the MCT is in this case finite for any p. On the other hand, for a given value of the resetting rate, the completion time is very sensitive to τ_0.

To sum up, sharp resetting is the most efficient resetting protocol in a well-known, static environment (search problem). However, such a protocol is highly sensitive to the timescale of the search problem and may easily fail in the case of an imperfect knowledge or fluctuating environment. If these fluctuations (or ignorance) are not too large and the average timescale is known, then constant-rate stochastic resets are advantageous. Otherwise, scale-free resets may be of great advantage, since they are robust to the timescale of the search problem.

4. Generalizations

The constant-rate and scale-free resetting protocols are special cases of the general family of resetting protocols of the form $r(t) = \alpha/t^\mu$, introduced in Section II B. As shown therein, supplementing diffusion with such a resetting protocol with $\mu < 1$ gives rise to the subdiffusive behavior. Thus, this protocol introduces a timescale and, for a fixed α, its MCT scales superlinearly with τ_0. It is therefore not robust, as the optimal α depends on τ_0. Nonetheless, this protocol may still be efficient in some search problems. Since in the limit of $\mu \to 0$ the standard constant-resetting case is recovered, in search problems with large variability one can expect that the fluctuations of the completion times are smaller for lower values of μ. Thus, the subdiffusive protocol may prove useful in the context of balancing the trade-off discussed in Sec. III D 3. The rich family of models with absolute-time-dependent stochastic resetting and the associated trade-off will be the subject of a separate study.

IV. CONCLUSIONS

We envision multiple applications of scale-free resetting, especially in optimization problems. For instance, it seems natural to ask about its relation to simulated annealing [57], and its applicability in evolutionary algorithms [6,7], deep learning via gradient methods [8,58], and biologically plausible learning techniques with three factors [59,60]. The robustness of the scale-free resetting protocol should offer additional benefits in the nonstationary setup of curriculum learning [61] and could potentially offer an explanation as to why aging [62,63] may be useful in learning. Of course the practical optimization problems are high-dimensional and thus the general renewal framework Eq. (8) should prove useful in the construction and analysis of the practical algorithms. Another interesting avenue of possible applications are models of evidence accumulation and decision making [64,65], as recently it was shown that stationary resets can modify splitting probabilities [66]. Restarts in this context may be interpreted as useful forgetting [67,68].

A number of open problems are left for future studies. To assess the efficiency of the proposed scheme, one could compare it to diffusion in time-dependent, scale-free potentials—such comparison in the case of static potentials and resets seems to favor the latter [54,55]. Moreover, it is important to find conditions under which scale-free resets can lower the expected completion time, similar to the simple criterion in the case of constant-rate resets [1,32,69]. A related question is how the optimal α depends on the details of a search problem at hand. Another important issue is the divergence of Eq. (2) at $t = 0$ which is infeasible and in practice a short-time cutoff has to be introduced. The optimal cut-off should depend on a cost associated with resets.

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