

Supplementary Material

An Analytical Result on the Assignment of Differential Dynamics to Different Types of Synapses in a Distributed Circuit: Conditions on Derivatives of $\mu_{mn}^*(x_n)$ at x^* . In this supplementary material, we simplify the model in order to analytically calculate whether facilitating or depressing connections are better to achieve homeostatic control at a steady state firing rate. In particular, we study the two-population mean-field model described by

$$\tau_m \frac{dx_m}{dt} = -x_m + f_m \left(v_m + \sum_{n=e,i} \mu_{mn}^*(x_n) x_n \right), \quad (\text{S1})$$

where $\vec{x} = \begin{pmatrix} x_e \\ x_i \end{pmatrix}$ are the firing rates of excitatory and inhibitory population, $\vec{v} = \begin{pmatrix} v_e \\ v_i \end{pmatrix}$ is the external input to excitatory and inhibitory population, the index $m = e, i$ means either excitatory or inhibitory population, and f_m is a monotonically increasing f-I curve of excitatory or inhibitory population. The dynamical synapses, $\mu_{mn}^*(x_n)$, are modeled to be a function of their pre-synaptic firing rate only, i.e. they are instantly equilibrating.

Let \vec{x}^* be a fixed point of the equation (S1), which satisfies the fixed point condition

$$x_m^* = f_m \left(v_m + \sum_{n=e,i} \mu_{mn}^*(x_n^*) x_n^* \right).$$

(S2)

Now, we apply perturbation theory and calculate how much an additional small current injection, $\delta\vec{v}$, to the system shifts the fixed point and how the dynamical synapses contribute to this shift. The shift in the output firing rates, $\delta\vec{x}$, is described by

$$x_m^* + \delta x_m = f_m \left(v_m + \delta v_m + \sum_{n=e,i} \mu_{mn}^* (x_n^* + \delta x_n) [x_n^* + \delta x_n] \right). \quad (\text{S3})$$

Linearization of equation (S3) around the old fixed point \vec{x}^* gives, up to the first order of $\delta \vec{x}$,

$$\delta x_m = \beta_m \left[\delta v_m + \sum_{n=e,i} (J_{mn} \delta x_n + d_{mn} \delta x_n x_n^*) \right], \quad (\text{S4})$$

where $J_{mn} = \mu_{mn}^* (x_n^*)$, $d_{mn} = d\mu_{mn}^* (x_n^*)/dx_n^*$ and we introduce the slope of f-I curve,

$$\beta_m = f_m' \left(v_m + \sum_{n=e,i} J_{mn} x_n^* \right),$$

evaluated at the old fixed point \vec{x}^* . Using some matrix notation $\beta = \begin{pmatrix} \beta_e & 0 \\ 0 & \beta_i \end{pmatrix}$,

$\tilde{J}_{mn} = \beta_m J_{mn}$, and $\tilde{D}_{mn} = \beta_m d_{mn} x_n^*$, we can solve for $\delta \vec{x}$ in equation (S4) in the following way:

$$\begin{aligned} \delta \vec{x} &= \beta \delta \vec{v} + (\tilde{J} + \tilde{D}) \delta \vec{x} \\ &= [I - (\tilde{J} + \tilde{D})]^{-1} \beta \delta \vec{v}. \end{aligned} \quad (\text{S5})$$

Therefore, the perturbations in the fixed point firing rates $\delta \vec{x}$, caused by the additional current injection $\delta \vec{v}$, are derived from equation (S5).

In the following, we study whether facilitating or depressing synapses are better to achieve smaller changes in the fixed point firing rate of the excitatory population, δx_e , compared with the static synapse case ($\tilde{D}_{mn} = 0$). To do so, we assume that the dynamic synapses are nearly static ($\tilde{D}_{mn} \approx 0$) and evaluate $\frac{\delta x_e}{\delta v_e}$ and $\frac{\delta x_e}{\delta v_i}$ up to the first order of

\tilde{D}_{mn} . After some calculations, we obtain

$$\begin{aligned} \frac{\delta x_e}{\delta v_e} &= \frac{\beta_e (1 + |\tilde{J}_{ii}|)}{\alpha} \left[1 + \frac{(1 + |\tilde{J}_{ii}|) \tilde{D}_{ee}}{\alpha} - \frac{|\tilde{J}_{ei}| |\tilde{J}_{ie} \tilde{D}_{ii}|}{\alpha (1 + |\tilde{J}_{ii}|)} - \frac{|\tilde{J}_{ei}| |\tilde{D}_{ie}|}{\alpha} + \frac{\tilde{J}_{ie} \tilde{D}_{ei}}{\alpha} \right] + O(\tilde{D}^2) \\ \frac{\delta x_e}{\delta v_i} &= \frac{-\beta_i |\tilde{J}_{ei}|}{\alpha} \left[1 + \frac{(1 + |\tilde{J}_{ii}|) \tilde{D}_{ee}}{\alpha} + \frac{(1 - \tilde{J}_{ee}) \tilde{D}_{ii}}{\alpha} - \frac{|\tilde{J}_{ei}| |\tilde{D}_{ie}|}{\alpha} - \frac{(1 - \tilde{J}_{ee})(1 + |\tilde{J}_{ii}|) \tilde{D}_{ei}}{\alpha |\tilde{J}_{ei}|} \right] + O(\tilde{D}^2), \end{aligned} \quad (\text{S5})$$

where $\alpha = (1 - \tilde{J}_{ee})(1 + |\tilde{J}_{ii}|) + |\tilde{J}_{ei}| |\tilde{J}_{ie}|$. Note that, from the monotonically increasing property of the f-I curves, $\beta_m > 0$; If we assume the stability of the old fixed point \bar{x}^* in the absence of the dynamical synapses, $\alpha > 0$. Hence, looking at equation (S5) term by term, in order to decrease the shift caused by δv_e , we obtain

$$\tilde{D}_{ee} < 0, \tilde{D}_{ii} > 0, \tilde{D}_{ie} > 0, \tilde{D}_{ei} < 0. \quad (\text{S6})$$

To decrease the shift caused by δv_i , we have the same condition,

$$\tilde{D}_{ee} < 0, \tilde{D}_{ii} > 0, \tilde{D}_{ie} > 0, \tilde{D}_{ei} < 0 \quad \text{if} \quad \tilde{J}_{ee} > 1, \quad (\text{S7})$$

but different condition

$$\tilde{D}_{ee} < 0, \tilde{D}_{ii} < 0, \tilde{D}_{ie} > 0, \tilde{D}_{ei} > 0 \quad \text{if} \quad \tilde{J}_{ee} < 1. \quad (\text{S8})$$

These conditions imply that the $E \rightarrow E$ connections should always be depressing and $E \rightarrow I$ connections should always be facilitating to have better homeostatic properties. On the other hand, the $I \rightarrow I$ connections should be depressing and the $I \rightarrow E$ connections should be facilitating if $\mu_{ee}^*(x_e^*) > 1/\beta_e$ (rephrased from $\tilde{J}_{ee} > 1$) but they should have the opposite properties if $\mu_{ee}^*(x_e^*) < 1/\beta_e$.

In the simulations shown in the main text we injected the same current to both the inhibitory and excitatory populations (i.e. $v_e = v_i = v$.) Additionally, we had $J_{ee} = J_{ie} = J_e$, $J_{ei} = J_{ii} = J_i$. We used the same FMS-surface, which is equivalent to using the same FI-curve, thus $f_e = f_i = f$, which implies $\beta_e = \beta_i = \beta$. In order to see

whether conditions (S7) or (S8) are correct for the networks in the main text we add term-by-term both equations in (S5) and examine the signs of each term in front of \tilde{D}_{mn} . After some calculation this gives to first order:

$$\frac{\delta x_e}{\delta v_e} + \frac{\delta x_e}{\delta v_i} = \left(\frac{\beta}{\alpha} \right) + \frac{\beta}{\alpha^2} \left((1 + |\tilde{J}_i|) \tilde{D}_{ee} - |\tilde{J}_i| \tilde{D}_{ii} - |\tilde{J}_i| \tilde{D}_{ie} + (1 + |J_i|) \tilde{D}_{ei} \right) \quad (\text{S8B})$$

which, based on the signs of the \tilde{D}_{mn} coefficients, shows that the $E \rightarrow E$ and $I \rightarrow I$ connections should be depressing and the $E \rightarrow I$ and $I \rightarrow E$ connections should be facilitating in order to have optimal homeostatic properties with respect to the parameters used in the main text.

A Theoretical Analysis of the Impact of Changes of Synaptic Weights on the Stability Properties of Networks with Static and Dynamic Synapses: Comparison of $\delta x_m / \delta J_{kl}$ for Static and Dynamic synapses We present an additional argument, which explains the stability of firing rates (in circuits of spiking neurons with dynamic synapses) with regard to changes in the scaling of synaptic weights. This stability was demonstrated in figure 10 panels B, C, E, F, H, and I, and it was shown in figure 10 panels A, D, G that the control circuits with static synapses do not share this stability property.

It can be easily seen that the previous analysis, which determined the conditions on the derivatives in order to achieve homeostatic properties of the excitatory firing rate with respect to a perturbation in the input current v_m , can be applied in a similar fashion to show that the same conditions achieve homeostatic properties of the excitatory population with respect to a perturbation δJ_{mn} in the synaptic weights $J_{mn} = \mu_{mn}^* (x_n^*)$. Said differently, the conditions given in the previous section provide networks with

dynamic synapses better homeostatic properties over the similar networks, with respect to synaptic weight perturbations. To see this we simply rewrite equation (S3) for a perturbation in synaptic strengths, which gives

$$x_m^* + \delta x_m = f_m \left(v_m + \sum_{n=e,i} (J_{mn} + \delta J_{mn} + d_{mn} \delta x_n) [x_n^* + \delta x_n] \right) \quad (\text{S9})$$

Linearization of equation (S9) around the old fixed point \bar{x}^* gives, up to the first order of $\delta \bar{x}$,

$$\delta x_m = \beta_m \left[\sum_{n=e,i} \delta J_{mn} x_n^* + \sum_{n=e,i} (J_{mn} \delta x_n + d_{mn} \delta x_n x_n^*) \right], \quad (\text{S10})$$

To see the similarity between equation (S10) and equation (S4) we set $\delta v_m = \sum_{n=e,i} \delta J_{mn} x_n^*$

and observe that $\delta x_m / \delta J_{kl} = \delta x_m / \delta v_k x_l^*$ (here we use indices k and l to denote all J , irrespective of neuronal population). This quantity $\delta x_m / \delta J_{kl}$ tells us how a small change in J_{kl} , δJ_{kl} , leads to a corresponding small change in the fixed points of the system, $\delta x_m = x_m - x_m^*$. We conclude that conditions for homeostatic properties for the perturbations $\delta x_m / \delta J_{kl}$ are derived from the homeostatic properties for the perturbations $\delta x_m / \delta v_m$, assuming again that the d_{mn} are small.

We show now that the heuristic proposed in the first section of the Results in the main text is appropriate to reduce a perturbation of x_e w.r.t. synaptic perturbations for the current based networks. The relevant assumptions are $v_e = v_i = v$, $J_{ee} = J_{ie} = J_e$, $J_{ei} = J_{ii} = J_i$ and $f_e = f_i = f$, which implies $\beta_e = \beta_i = \beta$. Observe that

$\frac{\delta x_e}{\delta J_e} = \frac{\delta x_e}{\delta J_{ee}} + \frac{\delta x_e}{\delta J_{ie}} = \left(\frac{\delta x_e}{\delta v_e} + \frac{\delta x_i}{\delta v_i} \right) x_e^*$, and similarly $\frac{\delta x_e}{\delta J_i} = \left(\frac{\delta x_e}{\delta v_e} + \frac{\delta x_i}{\delta v_i} \right) x_i^*$. These last

expressions are proportional to the two quantities in expression (S5) and so our previous analysis leading to (S8B) shows that heuristics used in the main paper are optimal for self-tuning of the excitatory population to perturbations in the synapse strengths J_e and J_i .

Finally, we provide in the following an example and show that $\delta x_m / \delta J_{kl}$ (now for a general δJ_{kl}) for model S1 with dynamic synapses is less than $\delta x_m / \delta J_{kl}$ for model (S1) with static synapses $d_{mn} = 0$. For further simplicity assume that in equation (S1) the FI-curves are identical and linear with slope β and that the $\mu_{mn}^*(x_n)$ are simply linear functions of the pre-synaptic firing rate written as $\mu_{mn}^*(x_n) = d_{mn}(x_n - x_n^*) + J_{mn}$ so that both systems have synaptic weights J_{kl} at the fixed point (x_e^*, x_i^*) . Under this assumption, we wish to calculate the quantity $\delta x_m / \delta J_{kl}$ for systems with static and dynamic synapses and then compare their absolute values. First we compute the fixed points of both models. These are

$$x_e^* = \beta v \frac{(1 + \beta |J_{ii}| - \beta |J_{ei}|)}{(1 - \beta J_{ee})(1 + \beta |J_{ii}|) + \beta^2 |J_{ei}| J_{ie}},$$

$$x_i^* = \beta v \frac{(1 - \beta J_{ee} + \beta J_{ie})}{(1 - \beta J_{ee})(1 + \beta |J_{ii}|) + \beta^2 |J_{ei}| J_{ie}}.$$

A first order perturbation around the fixed point allows us to calculate the desired quantities $\delta x_m / \delta J_{kl}$ for each model. We compute $\delta x_m / \delta J_{kl}$, using a first order expansion around \vec{x}^* and J :

$$\delta\bar{x} = [\mathbf{I} - (\tilde{\mathbf{J}} + \tilde{\mathbf{D}})]^{-1} \beta \delta J \bar{x}^*$$

$$\frac{\delta x_m}{\delta J_{kl}} = [\mathbf{I} - (\tilde{\mathbf{J}} + \tilde{\mathbf{D}})]^{-1}_{mk} \beta \delta x_l^*,$$

so we have everything we need now to calculate and compare the $\delta x_m / \delta J_{kl}$ values between the dynamic synapse model and the static synapse model ($\tilde{\mathbf{D}}$ has zeros only) for systematically varied values of J_{kl} , which is shown in figure S1. This example shows for the same perturbation in static synapse weights J_{kl} , the fixed point of model S1 with firing-rate dependent synaptic weights with conditions given by equations (S6) is perturbed less than the fixed point of the same model with static weights.

Figure S1 Firing-rate dependent synaptic weights confer robustness to perturbations in static synaptic strength. A system with pre-synaptic firing rate dependent synaptic weights is has an excitatory population that is more robust to perturbations of synaptic weights than the same system with static synapses. **A** Firing rates of system while varying the static synapse weight parameter J_{ee} (blue x_e^* , red x_i^*). **B** Perturbations of the static synapse strength J_{ee} lead to smaller changes in the firing rate for the system with firing rate dependent synaptic strengths. The computation of $\delta x_e / \delta J_{kl}$, for a system with static and dynamic synapses as a function of J_{ee} . The absolute value $|\delta x_e / \delta J_{kl}|$ is shown. The blue solid line shows the system with dynamic synapses and the dashed line shows the system with static synapses. **C, D** Same as A,B but for J_{ei} . **E, F** Same as A,B but for J_{ie} . **G, H** Same as A,B but for J_{ii} . All systems are locally stable in the ranges shown. ($J_{ee} = 10 \text{ pA/Hz}, |J_{ei}| = 20 \text{ pA/Hz}, J_{ie} = 10 \text{ pA/Hz}, |J_{ii}| = 20 \text{ pA/Hz}$). Each synaptic weight was individually varied between 0.8 and 1.2 times its base value while other synaptic weights were held constant. The derivatives were set to ($d_{ee} = -0.5 \text{ pA/Hz}^2, d_{ei} = -0.5 \text{ pA/Hz}^2, d_{ie} = 0.5 \text{ pA/Hz}^2, d_{ii} = 0.5 \text{ pA/Hz}^2$) and $\beta = 20 \text{ Hz/nA}$, with input $v_e = v_i = 1 \text{ nA}$.